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Abstract

In this paper we study stability properties of ridgelet and curvelet frames for mixed-smoothness Sobolev spaces with norm $\|f\|_s = \|f\|_{L_2(\mathbb{R}^d)} + \|s \cdot \nabla f\|_{L_2(\mathbb{R}^d)}$. Here $s \in \mathbb{S}^{d-1}$ is a transport direction and ∇ denotes the gradient of f. Such spaces arise as domains of linear, first order transport equations. The main result of this paper is that ridgelet frames are stable in $\|\cdot\|_s$ regardless of s, while curvelet frames are not. To show the second statement we explicitly construct functions f, g whose curvelet coefficients have all the same modulus but $\|f\|_s < \infty$ and $\|g\|_s = \infty$.

Keywords: Sobolev Spaces, Ridgelets, Curvelets, Transport Equations.
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Contents

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1 Introduction

Motivated by the fact that classical isotropic representation systems like wavelets do not perform well for high dimensional functions with singularities along hypersurfaces, a whole arsenal of new representation systems for $L_2(\mathbb{R}^d)$ have enriched the field of harmonic analysis in the last decades, specifically for the case $d = 2, 3$. To give an incomplete picture of these developments we only mention ridgelets [2], curvelets [4], contourlets [9], bandlets [16], shearlets [13], wedgelets [11],.... The main goal of all these representation systems is to properly handle data with anisotropic features.

A breakthrough in this direction has been obtained in [6] with the introduction of curvelets. There it is shown that one can construct a nonadaptive representation system that can approximate piecewise C^2-functions away from C^2 curves with an (almost) optimal rate in terms of nonlinear approximation.

Shortly afterwards shearlets have been introduced in [13]. They share the desirable properties of Curvelets with the additional advantage of a 'faithful digital transform', see e.g. [15] for details.

Curvelets have evolved from two ideas. The first one comes from the work of Hart Smith [19], who constructed curvelet-like systems to construct parametrices for hyperbolic PDEs. The second idea is based on a refinement of so-called ridgelets which are essentially ridge functions with oscillations across the ridge [2].

While most existing work on directional representation systems focuses on problems in image processing, essentially nothing is known regarding their applicability for solving operator equations in the spirit of [8].

In view of the superior treatment of anisotropic features by directional representation systems, it seems natural to aim at solving operator equations whose solutions typically exhibit anisotropic features.

A first (and essential) step in such a programme is to study stability properties of these systems with respect to the energy norm of the operator. For very simple transport problems this is carried out in the present paper where we investigate stability properties of curvelets and ridgelets in so-called mixed-smoothness Sobolev spaces, see below for the definition of these spaces.

1.1 Motivation

The theory of wavelets shows us that, in order to fully understand the properties of a representation system like curvelets or ridgelets, it is crucial to understand their approximation spaces. Not much is known in this direction for curvelets and ridgelets, we are only aware of the studies [2, 1]. However, if one wants to use e.g. a curvelet system for the solution of a partial differential equation, one usually needs to investigate stability properties in terms of certain Sobolev norms. To give an example, we consider the simple transport equation

$$s \cdot \nabla f + \kappa f = g,$$

where $s \in \mathbb{S}^{d-1}$, ∇ is the gradient of $f \in L_2(\mathbb{R}^d)$ and κ is (say) a Lipschitz function, bounded from above and below. Given g we would like to find f satisfying certain boundary conditions. Of course this is a very simple equation but in order to solve it e.g. with a

Petrov-Galerkin-type scheme, it is necessary to study the stability properties of our system of functions with respect to the norm $\|f\|_s^2 := \|f\|_2^2 + \|s \cdot \nabla f\|_2^2$. We refer to norms of this type as *mixed-smoothness Sobolev norm.* A more interesting equation arises if $f \in L_2(\mathbb{R}^d \times \mathbb{S}^{d-1})$ is also a function of s and s varies in $\mathbb{S}^{d-1}$. Equations of this type govern the radiation intensity in radiative transfer theory [18] and also arise in several other places. Clearly for such an equation, it would be desirable to have a representation system of functions that is simultaneously stable with respect to all angles $s \in \mathbb{S}^{d-1}$. In view of the directionality present in the norm $\|\cdot\|_s$ it is clear that a stable representation cannot be isotropic. For instance, wavelet systems are not stable with respect to the norm $\|\cdot\|_s$ for all s. Therefore, natural candidates for stable systems are given by curvelet or ridgelet systems.

1.2 Contributions

Our first result, Theorem 10, is that ridgelets satisfy the desired stability property. The second main result, Theorem 18, of this paper is that curvelets are not stable with respect to $\|\cdot\|_s$, regardless of s. We show the latter by giving explicit counterexamples for $d = 2$. In view of solving operator equations these results have two main implications:

- Ridgelet-based methods are promising candidates for developing Petrov-Galerkin-type solvers for transport problems, and
- Curvelet-based methods cannot be used to solve transport problems, at least with conventional Petrov-Galerkin schemes.

Our results also remain valid for more general norms of the following type: Given a finite sequence $(\mathbf{s}, \alpha) = (s_i, \alpha_i)_{i=1}^n$, $\alpha \in \mathbb{R}_+$, $s_i \in \mathbb{S}^{d-1}$ we define

$$\|f\|_{(\mathbf{s},\alpha)} := \|f\|_2 + \sum_{i=1}^{n} \| (s_i \cdot \nabla)^{\alpha_i} f\|_2,$$

where $(s_i \cdot \nabla)^{\alpha_i}$ should be interpreted in the sense of pseudodifferential calculus [14], see Definition 11. Theorem 12 says that ridgelets are also stable w.r.t. these more general norms.

1.3 Notation

We fix a dimension $d \geq 2$ with $d \in \mathbb{N}$. For two vectors $u, v \in \mathbb{R}^d$ we denote their inner product by $u \cdot v$. For $f \in L_1(\mathbb{R}^d) \cap L_2(\mathbb{R}^d)$ we denote by $\hat{f}$ its Fourier transform $\hat{f}(\xi) := \left(\frac{1}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} f(x) \exp(ix \cdot \xi) dx$ and extend this operation to $L_2(\mathbb{R}^d)$. We shall frequently use the notation $A \lesssim B$ to indicate that the quantity A is bounded by a constant times B. If $A \lesssim B$ and $B \lesssim A$ we shall also write $A \sim B$. For a function f we denote by ∇f its gradient. We will also use the notation $\delta_{jj'}$ for the Kronecker function which is one if $j = j'$ and zero otherwise. The symbol $B_{\mathbb{S}^{d-1}}(s, r)$ shall denote the geodesic ball of radius r in $\mathbb{S}^{d-1}$ around $s \in \mathbb{S}^{d-1}$. The symbol $|\cdot|$ will be used to denote the absolute value on $\mathbb{C}$, the Euclidean norm on $\mathbb{R}^d$ and the cardinality of a set.

2 Ridgelet Tight Frames

We start by constructing a tight frame of ridgelets for $L_2(\mathbb{R}^d)$. Recall that a system $(\psi_\lambda)_{\lambda\in\Lambda}$ of L_2 functions is called a *tight frame* of $L_2(\mathbb{R}^d)$ if

$$\|f\|_2^2 = \sum_{\lambda\in\Lambda} |\langle f, \psi_\lambda\rangle|^2 \quad \text{for all } f \in L_2(\mathbb{R}^d). \tag{1}$$

If (1) only holds with $\sim$ instead of $=$, we speak of a frame. The main property of a frame is that any $f \in L_2(\mathbb{R}^d)$ can be stably decomposed into, and reconstructed from the sequence $(\langle f, \psi_\lambda\rangle)_{\lambda\in\Lambda}$ – with some possible redundancy in the decomposition [7].

2.1 Preliminaries on Ridgelets

We begin by describing what we understand as ridgelets. In [2], a ridgelet transform has been introduced using a univariate (oscillatory) function ψ by mapping a function $f \in L_2(\mathbb{R}^d)$ to its transform coefficients

$$\langle f(x), a^{1/2}\psi\left(as\cdot x - t\right)\rangle, \quad s \in \mathbb{S}^{d-1}, t \in \mathbb{R}, a \in \mathbb{R}_+. \tag{2}$$

The function $x \mapsto a^{1/2}\psi\left(as\cdot x - t\right)$ is a *ridge function* (hence the name ridgelet) which only varies in the direction s. In particular this function is not in $L_2(\mathbb{R}^d)$ and therefore, (2), as it stands, makes no sense. For continuously varying parameters (a, s, t) one can still provide a stable reconstruction formula as it is the case with the Fourier transform but when we would like to discretize the parameters a, s, t e.g. to $a = 2^j, j \in \mathbb{N}$, $t \in \mathbb{Z}$ and s in some discrete and uniformly distributed subset of $\mathbb{S}^{d-1}$ with cardinality $\sim 2^j$, we need to evaluate (2) pointwise which makes no sense for general $f \in L_2(\mathbb{R}^d)$. Nevertheless, Candes showed in [2] that for compactly supported functions $f \in L_2([0,1]^d)$ a stable reconstruction can be given from the transform coefficients sampled on a discrete set. In other words for $f \in L_2([0,1]^d)$ discrete ridgelet frames can be constructed.

By relaxing the definition of a ridgelet a little, it is even possible to construct frames for $L_2(\mathbb{R}^d)$. The idea is that $\psi\left(as\cdot x - t\right)$ can also be written as $a^{1/2}\rho\left(D_a R_s x - t\right)$, where $D_a = \mathrm{diag}(a, 1, \dots, 1)$, R_s an orthogonal transform mapping $s \in \mathbb{S}^{d-1}$the vector $(1, 0, \dots, 0)$, $t \in \mathbb{R}^d$ and

$$\rho(x) = \psi(x_1). \tag{3}$$

If we allow the function ρ to vary also a little in the other coordinate directions besides $(1, 0, \dots, 0)$ so as to make $\rho \in L_2(\mathbb{R}^d)$, it can be shown that the parameters a, s, t can be sampled discretely to yield a frame for $L_2(\mathbb{R}^d)$. In this spirit, one might define a ridgelet system as a system of functions which are of the form

$$a^{1/2}\rho\left(D_a R_s x - t\right) \tag{4}$$

with some $\rho \in L_2(\mathbb{R}^d)$, which is oscillatory in the first coordinate, and for the parameters (a, s, t) ranging in some discrete set – typically $a = 2^j$, $j \in \mathbb{N}$, $t \in \mathbb{Z}^d$ and s in a uniformly distributed subset of $\mathbb{S}^{d-1}$ of cardinality $\sim 2^j$.

A yet more general viewpoint is to characterize ridgelets by their localization properties in space and frequency – without enforcing the rigid condition of being a frame of functions exactly of the form (4). This is the viewpoint that we shall take in this paper. For us, a ridgelet system is a system of functions that is adapted to the partitioning of frequency space outlined in Figure 1, left.

This partitioning consists of polar wedges with opening angle $\sim 2^{-j}$ and contained in the dyadic corona $2^j \leq |\xi| < 2^{j+1}$. An easy computation that takes in to account the oscillatory behaviour of ρ in the first coordinate, reveils that indeed the functions as defined in (4) have approximate frequency support in these wedges.

This viewpoint, which goes by the name of *decomposition spaces* [12], has been taken before for wavelet, Gabor, curvelet or shearlet systems.

The main advantage of this approach is that it allows for particularly simple tight frame constructions. On the other hand, the desirable approximation properties of the original definitions still remain valid in the more general context based on decomposition spaces.

It should be noted that yet another definition of ridgelets is given in [10].

2.2 A new Construction

In this section we present a novel construction of a ridgelet tight frame. For us, a ridgelet system is a system of functions which is adapted to the frequency tiling outlined in Figure 1 into angular wedges of opening angle $\sim 2^{-j}$ and height $\sim 2^j$, $j \in \mathbb{N}$. Therefore, the first step in our construction is to find a partition-of-unity which is adapted to this tiling. Since we are interested in discrete systems we need to find discrete sampling points on the sphere. The following lemma shows that one can always find reasonable uniformly distributed points on the sphere with a prescribed distance. A proof can be found e.g. in [1, Lemma 7]

Lemma 1. *Let $\mathbb{S}^{d-1}$ be the unit sphere equipped with the geodesic metric inherited from the Euclidean ambient space $\mathbb{R}^d$. Then for any $r > 0$ there exist $L \sim r^{-1}$, points $(s_l)_{l=1}^L$ on $\mathbb{S}^{d-1}$ and a constant A (independent of r) such that*

$$\bigcup_{l=1}^{L} B_{\mathbb{S}^{d-1}}(s_l, r) = \mathbb{S}^{d-1}, \tag{5}$$

and

$$\max_{l=1}^{L} |\{l' \neq l : B_{\mathbb{S}^{d-1}}(s_l, 2r) \cap B_{\mathbb{S}^{d-1}}(s_{l'}, 2r) \neq 0\}| \leq A. \tag{6}$$

In what follows we will construct a partition-of-unity for the ridgelet frequency tiling. We denote by e_1 the unit vector $(1, 0, \ldots, 0) \in \mathbb{S}^{d-1}$

Definition 2. *We fix smooth, nonnegative functions $V^{(j)} : \mathbb{S}^{d-1} \to \mathbb{R}$, $j \in \mathbb{N}$, $W : \mathbb{R}_+ \to \mathbb{R}$ and $W^{(0)} : \mathbb{R}_+ \to \mathbb{R}$ with the following properties:*

(i) supp $V^{(j)} \subset B_{\mathbb{S}^{d-1}}(e_1, 2 \cdot 2^{-j})$,

(ii) $V^{(j)}(s) \geq 1$ for all $s \in B_{\mathbb{S}^{d-1}}(e_1, 2^{-j})$,

(iii) $V^{(j)}(s) \leq 2$ for all $s \in B_{\mathbb{S}^{d-1}}(e_1, 2 \cdot 2^{-j})$,

(iv) supp $W \subset (1/2, 2)$,

(v) $W(r) \geq 1$ *for all* $r \in (3/4, 3/2)$*,*

(vi) *supp* $W^{(0)} \subset [0,2)$*,*

(vii) $W^{(0)}(r) \geq 1$ *for all* $r \leq 1$*.*

We define R_s as an orthogonal transform which maps the point $s \in \mathbb{S}^{d-1}$ to e_1. We also pick a sequence of sampling points $(s_{j,l})_{l=1}^{L_j}$ satisfying the conditions of Lemma 1 with $r = 2^{-j}$, $j \geq 1$.

Lemma 3. *There exist positive constants* C_1, C_2 *so that*

$$C_1 < \Phi(\xi) := \sum_{j,l} W(2^{-j}|\xi|)^2 V\left(R_{s_{j,l}}\frac{\xi}{|\xi|}\right)^2 + W^{(0)}(|\xi|)^2 < C_2 \text{ for all } \xi \in \mathbb{R}^d. \qquad (7)$$

Proof. We first show the existence of the constant C_1: since by (5), for any $\xi \in \mathbb{R}^d$ there exists j, l such that ξ has the representation $\left(|\xi|, \frac{\xi}{|\xi|}\right)$ with $|\xi| \in 2^j[3/4, 3/2]$ and $\frac{\xi}{|\xi|} \in B_{\mathbb{S}^{d-1}}(s_{j,l}, 2^{-j})$. From properties (ii), (v) it follows that

$$\Phi(\xi) \geq W(2^{-j}|\xi|)^2 V^{(j)}\left(\frac{\xi}{|\xi|}\right)^2 \geq 1$$

which gives the lower bound. The upper bound C_2 follows by noting that by (iii) and (6) any $x \in \mathbb{R}^d$ lies at most in the support of finitely many summands with bounded magnitude. □

The previous definitions enable us to finally define the frequency windows which are adapted to the ridgelet tiling.

Definition 4. *We define the following functions in terms of their Fourier transforms:*

$$\hat{\psi}_{j,l}(\xi) := \frac{W(2^{-j}|\xi|)V^{(j)}\left(R_{s_{j,l}}\frac{\xi}{|\xi|}\right)}{\sqrt{\Phi(\xi)}}, \quad j \geq 1,\ l = 1, \ldots, L_j,$$

and

$$\hat{\psi}_0(\xi) := \frac{W(|\xi|)}{\sqrt{\Phi(\xi)}}.$$

Observe that by (7) the division by Φ *is well-defined and the functions* $\psi_{j,l}$, ψ_0 *are in* $L_2(\mathbb{R}^d)$*.*

Definition 5. *We define the wedges*

$$P_0 := \{\xi : |\xi| \leq 2\}, \quad P_{j,l} := \left\{\xi : 2^{j-1} < |\xi| \leq 2^{j+1},\ \frac{\xi}{|\xi|} \in B_{\mathbb{S}^{d-1}}(s_{j,l}, 2^{-j}2)\right\}.$$

It follows that

$$\text{supp } \hat{\psi}_0 \subset P_0 \text{ and supp } \hat{\psi}_{j,l} \subset P_{j,l}.$$

We can now derive a semidiscrete representation formula for $L_2(\mathbb{R}^d)$.

Proposition 6. *We have*

$$\|f\|_2^2 = \|f * \psi_0\|_2^2 + \sum_{j,l} \|f * \psi_{j,l}\|_2^2. \tag{8}$$

Proof. The proof follows standard arguments, therefore we will ignore technical details like convergence issues. From the definition of $\psi_0, \psi_{j,l}$ and Φ it follows that

$$\hat{\psi}_0^2(\xi) + \sum_{j,l} \hat{\psi}_{j,l}(\xi)^2 = 1.$$

Therefore we have

$$\begin{aligned} \|f\|_2^2 &= \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 d\xi = \\ &= \int_{\mathbb{R}^d} \sum_{j,l} \hat{\psi}_{j,l}(\xi)^2 |\hat{f}(\xi)|^2 + \int_{\mathbb{R}^d} \psi_0(\xi)^2 |\hat{f}(\xi)|^2 d\xi \\ &= \|f * \psi_0\|_2^2 + \sum_{j,l} \|f * \psi_{j,l}\|_2^2. \end{aligned}$$

□

We go on to construct a tight frame decomposition of $L_2(\mathbb{R}^d)$ by discretizing the translational parameter in the convolutions in (8). To this end define the functions

$$\rho_{j,l,k}(x) := 2^{-j/2} T_{x_{j,l,k}} \psi_{j,k}, \quad \varphi_k := T_k \psi_0,$$

$k = (k_1, \dots, k_d) \in \mathbb{Z}^d$, $x_{j,l,k} := R^*_{s_{j,k}} (2^{-j} k_1, k_2, \dots, k_d)^T$, and $T_y f(\cdot) := f(\cdot - y)$.

Theorem 7. *The system*

$$(\varphi_k)_{k \in \mathbb{Z}^d} \cup (\rho_{j,k,l})_{j \geq 1, l \in [0, L_j], k \in \mathbb{Z}^d}$$

constitutes a tight frame for $L_2(\mathbb{R}^d)$.

Proof. In view of (8) we need to show that

$$\|f * \psi_{j,l}\|_2^2 = \sum_{k \in \mathbb{Z}^d} |\langle f, \rho_{j,l,k} \rangle|^2 \tag{9}$$

and

$$\|f * \psi_0\|_2^2 = \sum_{k \in \mathbb{Z}^d} |\langle f, \varphi_k \rangle|^2. \tag{10}$$

Since

$$\|f * \psi_{j,l}\|_2^2 = \int_{\mathbb{R}^2} |\hat{f}(\xi)| |\hat{\psi}_{j,l}(\xi)|^2 d\xi = \int_{P_{j,l}} |\hat{f}(\xi)| |\hat{\psi}_{j,l}(\xi)|^2 d\xi$$

equation (9) is shown by noting that the system $\left(2^{-j/2} \exp\left(i x_{j,l,k} \cdot \xi\right)\right)_{k \in \mathbb{Z}^d}$ constitutes an ONB of $L_2(P_{j,l})$ (compare also Lemma 4.2 in [6]). Equation (10) is proven in the same way. □

Remark 8. *Perhaps the best way to think of the functions $\rho_{j,l,k}$ is to write them as*

$$\rho_{j,l,k}(\cdot) := 2^{j/2} m^{(j,l,k)} \left(D_j R_{s_{l,k}} \cdot -k \right),$$

where $D_j = diag(2^j, 1, \ldots, 1)$. This way, the resemblance to (4) is seen. In our construction the functions $m^{(j,l,k)}$ are different for different indices but it is not difficult to show that this difference is only minor. To make this precise one would have to introduce the concept of ridgelet molecules *as has been done in [3] for curvelets and in [17] for wavelets (the latter construction goes by the name 'Vaguelettes'). In Figure 1 the localization properties of the ridgelet elements in space and frequency are depicted for the case $d = 2$.*

Remark 9. *In applications, where data is given as a discrete function defined on a digital grid, it is very inconvenient to work with the operation of rotation. For curvelets the same problem arises and a solution has been proposed with the introduction of so-called shearlets [15]. The main idea is to replace the rotation operations by appropreiate shear operations, the latter being also defined on digital data. These same adaptions can also be carried out for ridgelets.*

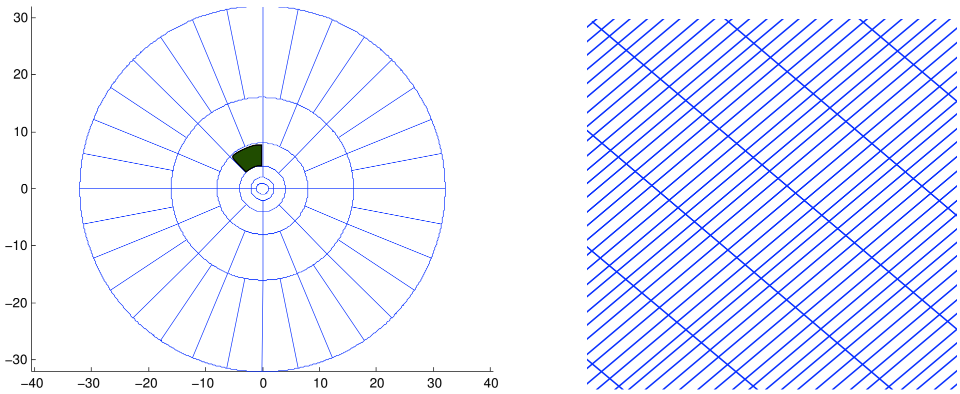

Figure 1: Left: Frequency space decomposition as indicated by the ridgelet frame. The essential support of $\hat{\psi}_{3,3}$ is colored. Right: Translational grid of the frame elements $\rho_{3,3,k}$, $k \in \mathbb{Z}^2$. The aspect ratio of the tiles is $1 \sim 2^{-3}$.

3 Stability Properties

3.1 Main Result

This section contains our first main result, namely the stability of the ridgelet tight frame with respect to the norm $\|\cdot\|_s$ as defined in the introduction. Our main stability theorem is as follows:

Theorem 10. *Let $s \in \mathbb{S}^{d-1}$. Then we have the norm equivalence*

$$\|f\|_2^2 + \|s \cdot \nabla f\|_2^2 \sim \sum_{k \in \mathbb{Z}^d} |\langle f, \varphi_k \rangle|^2 + \sum_{j,l} \left(1 + 2^{2j} |s \cdot s_{j,l}|^2\right) \sum_{k \in \mathbb{Z}^d} |\langle f, \rho_{j,l,k} \rangle|^2. \quad (11)$$

Proof. We have

$$\begin{aligned}
\|f\|_2^2 + \|s\cdot\nabla f\|_2^2 &\sim \int_{\mathbb{R}^d} \left(1+(s\cdot\xi)^2\right)|\hat{f}(\xi)|^2 d\xi \\
&= \int_{\mathbb{R}^2} \left(1+(s\cdot\xi)^2\right)|\hat{f}(\xi)|^2|\hat{\psi}_0(\xi)|^2 d\xi \\
&\quad + \sum_{j,l}\int_{\mathbb{R}^d} \left(1+(s\cdot\xi)^2\right)|\hat{f}(\xi)|^2|\hat{\psi}_{j,l}(\xi)|^2 d\xi \\
&= \int_{P_0} \left(1+(s\cdot\xi)^2\right)|\hat{f}(\xi)|^2|\hat{\psi}_0(\xi)|^2 d\xi \\
&\quad + \sum_{j,l}\int_{P_{j,l}} \left(1+(s\cdot\xi)^2\right)|\hat{f}(\xi)|^2|\hat{\psi}_{j,l}(\xi)|^2 d\xi \qquad (12)
\end{aligned}$$

The theorem is proven if we can show that

(i)

$$1+(s\cdot\xi)^2 \sim 1, \quad \xi \in P_0,$$

and

(ii)

$$1+(s\cdot\xi)^2 \sim 1 + 2^{2j}|s\cdot s_{j,l}|^2, \quad \xi\in P_{j,l},\ j\geq 1.$$

Equivalence (i) is a simple consequence of the Cauchy-Schwartz inequality. We now prove (ii). Let us split the set of indices j,l into

$$\mathcal{I}_0 := \left\{(j,l):\ |s\cdot s_{j,l}| \leq 2^{-j}20\right\}$$

and

$$\mathcal{I}_1 := \left\{(j,l):\ |s\cdot s_{j,l}| > 2^{-j}20\right\}.$$

Note that we always have

$$|s-s'| < \sigma(s,s'), \qquad (13)$$

σ denoting the geodesic metric in $\mathbb{S}^{d-1}$.

- We start with $(j,l)\in\mathcal{I}_0$. By our assumptions on (j,l) we then have

  $$1+2^{2j}|s\cdot s_{j,l}|^2 \sim 1. \qquad (14)$$

  On the other hand for any $\xi\in\mathbb{R}^d$ with representation $(|\xi|, s_\xi)$, Equation (13) together with the fact that

  $$s_\xi \in B_{\mathbb{S}^{d-1}}(s_{j,l}, 2^{-j}2)$$

  and the Cauchy-Schwartz inequality imply that

  $$|s\cdot s_{j,l} - s\cdot s_\xi| \lesssim 2^{-j},$$

  and therefore

  $$|s\cdot\xi| \sim 2^j|s\cdot s_\xi| \leq 2^j\left(|s\cdot s_{j,l}| + |s\cdot(s_\xi - s_{j,l})|\right) \lesssim 1.$$

This implies that for $\xi \in P_{j,l}$, $(j,l) \in \mathcal{I}_0$ we have

$$1 + (s \cdot \xi)^2 \sim 1 + 2^{2j}(s \cdot s_{j,l})^2 \sim 1$$

and that is (i).

- Now we let $(j,l) \in \mathcal{I}_1$ and $\xi \in P_{j,l}$ with spherical coordinates $(r, s_\xi) = \left(|\xi|, \frac{\xi}{|\xi|}\right)$, where $2^{j-1} < r < 2^{j+1}$ and $|s_\xi - s_{j,l}| \leq 2^{-j}2$. Consider

$$|s \cdot \xi| = r|s \cdot s_\xi| \sim 2^j |s \cdot s_\xi|.$$

We need to show that

$$|s \cdot s_\xi| \sim |s \cdot s_{j,l}|. \tag{15}$$

We have (note that the division by $s \cdot s_{j,l}$ is permitted since $s \cdot s_{j,l} \neq 0$ by the assumption that $(j,l) \in \mathcal{I}_1$)

$$|s \cdot s_\xi| = |s \cdot s_{j,l} + s \cdot (s_\xi - s_{j,l})| = |s \cdot s_{j,l}| \left| 1 + \frac{s \cdot (s_\xi - s_{j,l})}{s \cdot s_{j,l}} \right|$$

and therefore it remains to bound the quantity

$$\left| 1 + \frac{s \cdot (s_\xi - s_{j,l})}{s \cdot s_{j,l}} \right|$$

from above and below. We start with the estimate from below:

$$\begin{aligned} \left| 1 + \frac{s \cdot (s_\xi - s_{j,l})}{s \cdot s_{j,l}} \right| &\geq 1 - \left| \frac{s \cdot (s_\xi - s_{j,l})}{s \cdot s_{j,l}} \right| \\ &\geq 1 - \frac{2^{-j}2}{2^{-j}20} \geq 9/10. \end{aligned}$$

On the other hand we have

$$\begin{aligned} \left| 1 + \frac{s \cdot (s_\xi - s_{j,l})}{s \cdot s_{j,l}} \right| &\leq 1 + \left| \frac{s \cdot (s_\xi - s_{j,l})}{s \cdot s_{j,l}} \right| \\ &\leq 11/10. \end{aligned}$$

This shows (15) and therefore we have

$$1 + (s \cdot \xi)^2 \sim 1 + 2^{2j}(s \cdot s_{j,l})^2,$$

which is (ii).

In view of (12), (9) and (10) we have

$$\begin{aligned} \|f\|_2^2 + \|s \cdot \nabla f\|_2^2 &\sim \int_{P_0} \left(1 + (s \cdot \xi)^2\right) |\hat{f}(\xi)|^2 |\hat{\psi}_0(\xi)|^2 d\xi \\ &\quad + \sum_{j,l} \int_{P_{j,l}} \left(1 + (s \cdot \xi)^2\right) |\hat{f}(\xi)|^2 |\hat{\psi}_{j,l}(\xi)|^2 d\xi \\ &\sim \|f * \psi_0\|_2^2 + \sum_{j,l} \left(1 + 2^{2j}(s \cdot s_{j,l})^2\right) \|f * \psi_{j,l}\|_2^2 \\ &= \sum_k |\langle f, \varphi_k \rangle|^2 + \sum_{j,l,k} \left(1 + 2^{2j}(s \cdot s_{j,l})^2\right) |\langle f, \rho_{j,l,k} \rangle|^2. \end{aligned}$$

This proves the theorem. □

3.2 More general Spaces

It is perhaps worth noting that ridgelet systems provide stable decompositions for a whole scale of spaces which we call mixed-smoothness Sobolev spaces. The definition is as follows:

Definition 11. *For $\alpha > 0$ we define the operator $(s \cdot \nabla)^{\alpha}$ acting on tempered distributions via*

$$\hat{f} \mapsto (s \cdot \xi)^{\alpha} \hat{f}.$$

For a finite sequence $(\mathbf{s}, \alpha) := (s_i, \alpha_i)_{i=1}^n \in \left(\mathbb{S}^{d-1} \times \mathbb{R}_+\right)^n$ we define the norm

$$\|f\|_{(\mathbf{s},\alpha)} := \|f\|_2 + \sum_{i=1}^n \| (s_i \cdot \nabla)^{\alpha_i} f\|_2.$$

We have the following generalization of Theorem 10:

Theorem 12. *Given $(\mathbf{s}, \alpha) \in \left(\mathbb{S}^{d-1} \times \mathbb{R}_+\right)^n$ we have the norm equivalence*

$$\|f\|_{(\mathbf{s},\alpha)}^2 \sim \sum_k |\langle f, \varphi_k\rangle|^2 + \sum_{j,l,k} \left(1 + \sum_{i=1}^n 2^{2\alpha_i j} (s_i \cdot s_{j,l})^{2\alpha_i}\right) |\langle f, \rho_{j,l,k}\rangle|^2.$$

The implicit constant depends on n.

Proof. The proof is virtually identical to the proof of Theorem 10 and therefore we omit it. □

4 Curvelets do *not* provide stable Decompositions for mixed Smoothness Sobolev Spaces

In this section we prove the second main result, namely that curvelets are not stable with respect to $\| \cdot \|_s$, regardless of $s \in \mathbb{S}^{d-1}$. We construct our specific counterexample for $d = 2$ but a modification along the same lines to arbitrary d is possible, albeit with more notational overload.

In order to show the instability result, we construct two functions f and g in $L(\mathbb{R}^2)$ whose coefficients all have the same magnitude in a curvelet tight frame but $\|f\|_s < \infty$ and $\|g\|_s = \infty$. The idea for this construction is depicted in Figure 2.

In what follows we shall identify $\mathbb{S}^1$ with the interval $[-\pi, \pi)$.

First we give a construction of a curvelet tight frame convenient for our purposes. Similar to above we start with two window functions.

Definition 13. *We define univariate, nonnegative C^{∞} functions $W_c(r)$, $V_c(t)$ so that*

(i) supp $W_c \subset \left(\frac{3}{4}, 4\right)$,

(ii) supp $V_c \subset \left(-\frac{3}{4}, \frac{3}{4}\right)$,

(iii) $\sum_{j\in\mathbb{Z}} W_c^2 (4^{-j} r) = 1$, for all $r \in \mathbb{R}$ and

(iv) $\sum_{l\in\mathbb{Z}} V_c^2(t-l) = 1$ *for all* $t \in \mathbb{R}$.

The construction of such window functions is standard in wavelet theory. For the convenience of the reader we sketch the construction of W_c: Start with any smooth nonnegative function $\widetilde{W}$ which is supported in $[\frac{3}{4}, 4]$ and strictly positive on $(\frac{3}{4}, 4)$. Then define

$$W_c(r) := \frac{\widetilde{W}(r)}{\left(\sum_{l\in\mathbb{Z}} \widetilde{W}^2(4^{-l}r)\right)^{1/2}}.$$

This function satisfies (i). It also satisfies (iii), since

$$\sum_{j\in\mathbb{Z}} W_c^2(4^{-j}r) = \sum_{j\in\mathbb{Z}} \frac{\widetilde{W}^2(4^{-j}r)}{\sum_{l\in\mathbb{Z}} \widetilde{W}^2(4^{-l}4^{-j}r)} = \frac{\sum_{j\in\mathbb{Z}} \widetilde{W}^2(4^{-j}r)}{\sum_{l\in\mathbb{Z}} \widetilde{W}^2(4^{-l}r)} = 1.$$

The construction of V_c is similar. Observe that in view of (i) and (iii) resp. (ii) and (iv) we have

$$W(r) = 1 \text{ for } r \in (1,3) \text{ and } V(t) = 1 \text{ for } t \in (-\frac{1}{4}, \frac{1}{4}).$$

Now, similar to the rigelet definitions above we define

$$s_{j,l} := 2\pi l 2^{-j}, \quad L_j := 2^j - 1$$

and make the following definition:

Definition 14. *We write* r, ω *for the polar variables of the frequency plane and define functions*

$$\hat{\varphi}^c(r,\omega)^2 := \sum_{j=0}^{-\infty} \sum_{l=0}^{L_j} W_c\left(4^{-j}r\right)^2 V_c\left(\frac{2^j}{2\pi}\left(\omega - s_{j,l}\right)\right)^2, \tag{16}$$

and

$$\hat{\psi}_{j,l}^c(r,\omega) := W_c\left(4^{-j}r\right) V_c\left(\frac{2^j}{2\pi}\left(\omega - s_{j,l}\right)\right), \tag{17}$$

where $j > 0$.

Sampling the translational variable on the integer grid yields the following definition.

Definition 15. *We define for* $k = (k_1, k_2) \in \mathbb{Z}^2$ *the functions*

$$\varphi_k^c(\cdot) := \varphi^c(\cdot - k),$$

$$\gamma_{j,0,k}(\cdot) := 2^{-3j/2}\psi_{j,l}^c\left(\cdot - (4^{-j}k_1, 2^{-j}k_2)\right),$$

and

$$\gamma_{j,l,k}(\cdot) := \gamma_{j,0,k}\left(R_{s_{j,l}}\cdot\right).$$

The same arguments as above for the ridgelet case yield

Theorem 16. *The system* $(\varphi_k^c)_{k\in\mathbb{Z}^2} \cup (\gamma_{j,l,k})_{j>0, l=0,\ldots,L_j, k\in\mathbb{Z}^2}$ *constitutes a tight frame for* $L_2(\mathbb{R}^2)$.

Definition 17. *We call this tight frame a* curvelet frame.

Observe that, unlike ridgelets, curvelets are supported in frequency rectangles of aspect ratio $\sim 4^j \times 2^j$. This property is called parabolic scaling in the literature. The parabolic scaling allows curvelets to be well-localized in space and therefore better suited for applications where it is important to approximate curved singularities [5]. However, as we shall see, the increase in angular uncertainty of curvelets compared to ridgelets causes instability of the curvelet frame in mixed-smoothess Sobolev spaces. Indeed we will show the following:

Theorem 18. *There exist two functions $f, g \in L_2(\mathbb{R}^2)$ such that*

(i)

$$\|e_1 \cdot \nabla f\|_2 < \infty,$$

(ii)

$$\|e_1 \cdot \nabla g\|_2 = \infty, \text{ and}$$

(iii)

$$|\langle f, \gamma_{j,l,k}\rangle| = |\langle g, \gamma_{j,l,k}\rangle| \text{ and } |\langle f, \varphi_k^c\rangle| = |\langle g, \varphi_k^c\rangle| \quad \text{for all indices } j, l, k.$$

Proof. First we define the rectangles

$$Q_j := \left[-\frac{1}{2}2^j, \frac{1}{2}2^j\right] \times \left[4^j, 24^j\right].$$

We have the following property for $j > 2$:

$$\hat{\psi}_{j,l}^c \chi_{Q'_j} = \delta_{jj'}\delta_{l2^{j-2}}. \tag{18}$$

Indeed this follows immediately from the definition of $\hat{\psi}_{j,l}^c$. Here, $\chi_{Q_{j'}}$ denotes the characteristic function of $Q_{j'}$. Intuitively, equation (18) simply means that $Q_{j'}$ only intersects with the support of $\hat{\psi}_{j',2^{j'-2}}^c$. Therefore, by the partition of unity property of the functions $\hat{\varphi}^c, \hat{\psi}_{j,l}^c$ it follows that the function $\hat{\psi}_{j',2^{j'-2}}^c$ restricted to $Q_{j'}$ must equal 1. Also observe that $s_{j',2^{j'-2}} = \pi/2$. Now we are ready to define the functions f and g. Let $0 < \varepsilon < 4$.

$$\hat{f} := \sum_{j>2} 2^{-j-\varepsilon j} \chi_{[0,1]\times[4^j, 24^j]} \tag{19}$$

and

$$\hat{g} := \sum_{j>2} 2^{-j-\varepsilon j} \chi_{[\frac{1}{2}2^j - 1, \frac{1}{2}2^j]\times[4^j, 24^j]}. \tag{20}$$

It follows that

$$\|f\|_2^2 = \sum_{j>2} 4^{-j-\varepsilon j} 4^j < \infty$$

and

$$\|e_1 \cdot \nabla f\|_2^2 \leq \sum_{j>2} 4^{-j-\varepsilon j} 4^j < \infty.$$

On the other hand,

$$\|g\|_2^2 = \sum_{j>2} 4^{-j-\varepsilon j} 4^j < \infty$$

but

$$\|e_1 \cdot \nabla g\|_2^2 \geq \sum_{j>2} 4^{-j-\varepsilon j} \frac{1}{4} 4^j 4^j = \infty.$$

Now, let us inspect the curvelet coefficients of f. Observe that

$$[0,1] \times \left[4^j, 24^j\right] \subset Q_j \text{ and } \left[\frac{1}{2}2^j - 1, \frac{1}{2}2^j\right] \times \left[4^j, 24^j\right] \subset Q_j. \tag{21}$$

We have for $j > 2$ (for $j \leq 2$ we have $\langle f, \gamma_{j,l,k}\rangle = 0$ and also $\langle f, \varphi_k^c\rangle = 0$)

$$\begin{aligned}
\langle f, \gamma_{j,l,k}\rangle &= 2^{-3j/2} \int_{\mathbb{R}^2} \hat{f}(\xi) \hat{\psi}_{j,l}^c(\xi) \exp\left(i R_{s_{j,l}} (4^{-j}k_1, 2^{-j}k_2)^t \cdot \xi\right) d\xi \\
&= \delta_{l 2^{j-2}} 2^{-3j/2} 2^{-j-j\varepsilon} \int_{[0,1]\times[4^j,24^j]} \psi_{j,l}^c(\xi) \exp\left(i(4^{-j}k_1\xi_2 + 2^{-j}k_2\xi_1)\right) d\xi \\
&= \delta_{l 2^{j-2}} 2^{-3j/2} 2^{-j-j\varepsilon} \int_{[0,1]\times[4^j,24^j]} \exp\left(i(4^{-j}k_1\xi_2 + 2^{-j}k_2\xi_1)\right) d\xi,
\end{aligned}$$

where the last two equalities make use of (18) and (21). Similarily, we compute

$$\begin{aligned}
\langle g, \gamma_{j,l,k}\rangle &= \delta_{l 2^{j-2}} 2^{-3j/2} 2^{-j-j\varepsilon} \int_{[\frac{1}{2}2^j-1, \frac{1}{2}2^j]\times[4^j,24^j]} \exp\left(i(4^{-j}k_1\xi_2 + 2^{-j}k_2\xi_1)\right) d\xi \\
&= \delta_{l 2^{j-2}} 2^{-3j/2} 2^{-j-j\varepsilon} \int_{[0,1]\times[4^j,24^j]} \exp\left(i(4^{-j}k_1\xi_2 + 2^{-j}k_2(\xi_1 + \frac{1}{2}2^j - 1))\right) d\xi \\
&= \exp(ik_2(1/2 - 2^{-j}))\langle f, \gamma_{j,l,k}\rangle.
\end{aligned}$$

In particular this implies (iii), which is what we sought. □

5 Conclusion

In this paper we studied stability properties of curvelet and ridgelet frames in mixed-smoothness Sobolev spaces. It turns out that curvelets are not suitable to characterize such spaces, while ridgelets are. It is straightforward to adapt our results to other systems based on parabolic scaling like for instance the shearlet transform.

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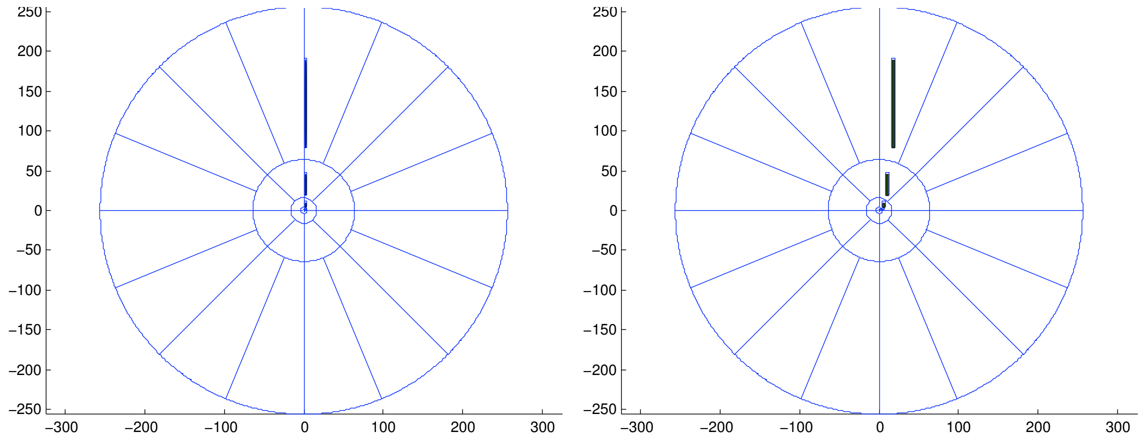

Figure 2: Left: Illustration of the frequency support of the function f which satisfies $\|e_1 \cdot \nabla f\|_2 < \infty$. Right: Illustration of the frequency support of the function g with $\|e_1 \cdot \nabla g\|_2 = \infty$. The curvelet decomposition of the frequency plane cannot distinguish between these two functions. Note that in this figure, as opposed to Figure 1, the aspect ratio of the angular wedges is $4^j \sim 2^j$.

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